

# Elliptic Curves

(PARI-GP version 2.17.0)

An elliptic curve is initially given by 5-tuple  $v = [a_1, a_2, a_3, a_4, a_6]$  attached to Weierstrass model or simply  $[a_4, a_6]$ . It must be converted to an *ell* struct.

Initialize *ell* struct over domain  $D$       **E = ellinit**( $v, \{D = 1\}$ )  
over **Q**       $D = 1$   
over **F<sub>p</sub>**       $D = p$   
over **F<sub>q</sub>**,  $q = p^f$        $D = \text{ffgen}([p, f])$   
over **Q<sub>p</sub>**, precision  $n$        $D = O(p^n)$   
over **C**, current bitprecision       $D = 1.0$   
over number field  $K$        $D = nf$

Points are [x,y], the origin is [0]. Struct members accessed as **E.member**:

- All domains: **E.a1,a2,a3,a4,a6, b2,b4,b6,b8, c4,c6, disc, j**
- $E$  defined over **R** or **C**
  - $x$ -coords. of points of order 2      **E.roots**
  - periods / quasi-periods      **E.omega, E.eta**
  - volume of complex lattice      **E.area**
- $E$  defined over **Q<sub>p</sub>**
  - residual characteristic      **E.p**
  - If  $|j|_p > 1$ : Tate's  $[u^2, u, q, [a, b], \mathcal{L}]$       **E.tate**
- $E$  defined over **F<sub>q</sub>**
  - characteristic      **E.p**
  - $\#E(\mathbf{F}_q)/\text{cyclic structure/generators}$       **E.no, E.cyc, E.gen**
- $E$  defined over **Q**
  - generators of  $E(\mathbf{Q})$  (require **elldata**)      **E.gen**
  - $[a_1, a_2, a_3, a_4, a_6]$  from  $j$ -invariant      **ellfromj**( $j$ )
  - cubic/quartic/biquadratic to Weierstrass      **ellfromeqn**( $eq$ )
  - add points  $P + Q$  /  $P - Q$       **elladd**( $E, P, Q$ ), **ellsub**
  - negate point      **ellneg**( $E, P$ )
  - compute  $n \cdot P$       **ellmul**( $E, P, n$ )
  - sum of Galois conjugates of  $P$       **elltrace**( $E, P$ )
  - check if  $P$  is on  $E$       **ellisoncurve**( $E, P$ )
  - order of torsion point  $P$       **ellorder**( $E, P$ )
  - $y$ -coordinates of point(s) for  $x$       **ellordinate**( $E, x$ )
  - $[\wp(z), \wp'(z)] \in E(\mathbf{C})$  attached to  $z \in \mathbf{C}$       **ellztopoint**( $E, z$ )
  - $z \in \mathbf{C}$  such that  $P = [\wp(z), \wp'(z)]$       **ellpointtoz**( $E, P$ )
  - $z \in \bar{\mathbf{Q}}^*/q^{\mathbf{Z}}$  to  $P \in E(\bar{\mathbf{Q}}_p)$       **ellztopoint**( $E, z$ )
  - $P \in E(\bar{\mathbf{Q}}_p)$  to  $z \in \bar{\mathbf{Q}}^*/q^{\mathbf{Z}}$       **ellpointtoz**( $E, P$ )
- **Change of Weierstrass models, using**  $v = [u, r, s, t]$ 
  - change curve  $E$  using  $v$       **ellchangecurve**( $E, v$ )
  - change point  $P$  using  $v$       **ellchangepoint**( $P, v$ )
  - change point  $P$  using inverse of  $v$       **ellchangepointinv**( $P, v$ )
  - is  $E$  isomorphic to  $F$ ?      **ellsisom**( $E, F$ )
- **Twists and isogenies**
  - quadratic twist      **elltwist**( $E, d$ )
  - $n$ -division polynomial  $f_n(x)$       **elldivpol**( $E, n, \{x\}$ )
  - $[n]P = (\phi_n \psi_n : \omega_n : \psi_n^3)$ ; return  $(\phi_n, \psi_n^2)$       **ellxn**( $E, n, \{x\}$ )
  - isogeny from  $E$  to  $E/G$       **ellisogeny**( $E, G$ )
  - apply isogeny to  $g$  (point or isogeny)      **ellisogenyapply**( $f, g$ )
  - torsion subgroup with generators      **elltors**( $E$ )
- **Formal group**
  - formal exponential,  $n$  terms      **ellformalexp**( $E, \{n\}, \{x\}$ )
  - formal logarithm,  $n$  terms      **ellformalog**( $E, \{n\}, \{x\}$ )
  - $\log_E(-x(P)/y(P)) \in \mathbf{Q}_p$ ;  $P \in E(\mathbf{Q}_p)$       **ellpadiclog**( $E, p, n, P$ )
  - $P$  in the formal group      **ellformalpoint**( $E, \{n\}, \{x\}$ )
  - $[\omega/dt, x\omega/dt]$       **ellformaldifferential**( $E, \{n\}, \{x\}$ )

$w = -1/y$  in parameter  $-x/y$       **ellformalw**( $E, \{n\}, \{x\}$ )

## Curves over finite fields, Pairings

random point on  $E$       **random**( $E$ )  
 $\#E(\mathbf{F}_q)$       **ellcard**( $E$ )  
 $\#E(\mathbf{F}_q)$  with almost prime order      **ellsea**( $E, \{tors\}$ )  
structure  $\mathbf{Z}/d_1\mathbf{Z} \times \mathbf{Z}/d_2\mathbf{Z}$  of  $E(\mathbf{F}_q)$       **ellgroup**( $E$ )  
is  $E$  supersingular?      **ellissupersingular**( $E$ )  
random supersingular  $j$ -invariant over  $F_2^2$       **ellsupersingularj**( $p$ )  
Weil pairing of  $m$ -torsion pts  $P, Q$       **ellweilpairing**( $E, P, Q, m$ )  
Tate pairing of  $P, Q$ ;  $P$   $m$ -torsion      **elltatepairing**( $E, P, Q, m$ )  
Discrete log, find  $n$  s.t.  $P = [n]Q$       **elllog**( $E, P, Q, \{ord\}$ )

## Curves over Q

### Reduction, minimal model

minimal model of  $E/\mathbf{Q}$       **ellminimalmodel**( $E, \{\&v\}$ )  
quadratic twist of minimal conductor      **ellminimaltwist**( $E$ )  
 $[k]P$  with good reduction      **ellnonsingularmultiple**( $E, P$ )  
 $E$  supersingular at  $p$ ?      **ellissupersingular**( $E, p$ )  
affine points of naïve height  $\leq h$       **ellratpoints**( $E, h$ )

### Complex heights

canonical height of  $P$       **ellheight**( $E, P$ )  
canonical bilinear form taken at  $P, Q$       **ellheight**( $E, P, Q$ )  
height regulator matrix for pts in  $L$       **ellheightmatrix**( $E, L$ )

### $p$ -adic heights

cyclotomic  $p$ -adic height of  $P \in E(\mathbf{Q})$       **ellpadicheight**( $E, p, n, P$ )  
... bilinear form at  $P, Q \in E(\mathbf{Q})$       **ellpadicheight**( $E, p, n, P, Q$ )  
... matrix at vector for pts in  $L$       **ellpadicheightmatrix**( $E, p, n, L$ )  
... regulator for canonical height      **ellpadicregulator**( $E, p, n, Q$ )  
Frobenius on  $\mathbf{Q}_p \otimes H_{dR}^1(E/\mathbf{Q})$       **ellpadicfrobenius**( $E, p, n$ )  
slope of unit eigenvector of Frobenius      **ellpadics2**( $E, p, n$ )

### Isogenous curves

matrix of isogeny degrees for **Q**-isog. curves      **ellisomat**( $E$ )  
tree of prime degree isogenies      **ellisotree**( $E$ )  
a modular equation of prime degree  $N$       **ellmodulareqn**( $N$ )

### $L$ -function

$p$ -th coeff  $a_p$  of  $L$ -function,  $p$  prime      **ellap**( $E, p$ )  
 $k$ -th coeff  $a_k$  of  $L$ -function      **ellak**( $E, k$ )  
 $L(E, s)$  (using less memory than **lfun**)      **elllseries**( $E, s$ )  
 $L^{(r)}(E, 1)$  (using less memory than **lfun**)      **elll1**( $E, r$ )  
a Heegner point on  $E$  of rank 1      **ellheegner**( $E$ )  
order of vanishing at 1      **ellanalyticrank**( $E, \{eps\}$ )  
root number for  $L(E, \cdot)$  at  $p$       **ellrootno**( $E, \{p\}$ )  
modular parametrization of  $E$       **elltaniyama**( $E$ )  
degree of modular parametrization      **ellmoddegree**( $E$ )  
compare with  $H^1(X_0(N), \mathbf{Z})$  (for  $E' \rightarrow E$ )      **ellweilcurve**( $E$ )  
Manin constant of  $E$       **ellmaninconstant**( $E$ )

$p$ -adic  $L$  function  $L_p^{(r)}(E, d, \chi^s)$       **ellpadicL**( $E, p, n, \{s\}, \{r\}, \{d\}$ )  
BSD conjecture for  $L_p^{(r)}(E_D, \chi^0)$       **ellpadicbsd**( $E, p, n, \{D = 1\}$ )  
Iwasawa invariants for  $L_p(E_D, \tau^i)$       **ellpadicIamdamu**( $E, p, D, i$ )

### Rational points

attempt to compute  $E(\mathbf{Q})$       **ellrank**( $E, \{effort\}, \{points\}$ )  
initialize for later **ellrank** calls,      **ellrankinit**( $E$ )  
saturate  $\langle P_1, \dots, P_n \rangle$  wrt. primes  $\leq B$       **ellsaturation**( $E, P, B$ )  
2-covers of the curve  $E$       **ell2cover**( $E$ )

### Elldata package, Cremona's database:

db code "11a1"  $\leftrightarrow$  [conductor, class, index]      **ellconvertname**( $s$ )  
generators of Mordell-Weil group      **ellgenerators**( $E$ )  
look up  $E$  in database      **ellidentify**( $E$ )

all curves matching criterion      **ellsearch**( $N$ )  
loop over curves with cond. from  $a$  to  $b$       **forell**( $E, a, b, seq$ )

## Curves over number field $K$

coeff  $a_p$  of  $L$ -function      **ellap**( $E, p$ )  
Kodaira type of  $p$ -fiber of  $E$       **elllocalred**( $E, p$ )  
integral model of  $E/K$       **ellintegralmodel**( $E, \{\&v\}$ )  
minimal model of  $E/K$       **ellminimalmodel**( $E, \{\&v\}$ )  
minimal discriminant of  $E/K$       **ellminimaldisc**( $E$ )  
cond, min mod, Tamagawa num  $[N, v, c]$       **ellglobalred**( $E$ )  
global Tamagawa number      **elltamagawa**( $E$ )  
test if  $E$  has CM      **elliscm**( $E$ )  
 $P \in E(K)$   $n$ -divisible?  $[n]Q = P$       **ellisdivisible**( $E, P, n, \{\&Q\}$ )

### $L$ -function

A domain  $D = [c, w, h]$  in initialization mean we restrict  $s \in \mathbf{C}$  to domain  $|\Re(s) - c| < w, |\Im(s)| < h$ ;  $D = [w, h]$  encodes  $[1/2, w, h]$  and  $[h]$  encodes  $D = [1/2, 0, h]$  (critical line up to height  $h$ ).  
vector of first  $n$   $a_k$ 's in  $L$ -function      **ellan**( $E, n$ )  
init  $L^{(k)}(E, s)$  for  $k \leq n$       **L = lfunit**( $E, D, \{n = 0\}$ )  
compute  $L(E, s)$  ( $n$ -th derivative)      **lfun**( $L, s, \{n = 0\}$ )  
 $L(E, 1, r)/(r! \cdot R \cdot \#Sha)$  assuming BSD      **ellbsd**( $E$ )

## Other curves of small genus

A hyperelliptic curve  $C$  is given by a pair  $[P, Q]$  ( $y^2 + Qy = P$  with  $Q^2 + 4P$  squarefree) or a single squarefree polynomial  $P$  ( $y^2 = P$ ).  
check if  $[x, y]$  is on  $C$       **hyperellisoncurve**( $C, [x, y]$ )  
 $y$ -coordinates of point(s) for  $x$       **hyperellordinate**( $C, x$ )  
discriminant of  $C$       **hyperelldisc**( $C$ )  
Cremona-Stoll reduction      **hyperellred**( $C$ )  
apply  $m = [e, [a, b; c, d], H]$  to model      **hyperellchangecurve**( $C, m$ )  
minimal discriminant of integral  $C$       **hyperellminimaldisc**( $C$ )  
minimal model of integral  $C$       **hyperellminimalmodel**( $C$ )  
reduction of  $y^2 + Qy = P$  (genus 2)      **genus2red**( $C, \{p\}$ )  
Igusa invariants for  $C$  of genus 2      **genus2igusa**( $C$ )  
affine rational points of height  $\leq h$       **hyperellratpoints**( $C, h$ )  
find a rational point on a conic,  ${}^tXGX = 0$       **qfsolve**( $G$ )  
 $[H, U]$  such that  $H = c^tUGU$  has minimat  $d$       **qfminimize**( $G$ )  
quadratic Hilbert symbol (at  $p$ )      **hilbert**( $x, y, \{p\}$ )  
all solutions in  $\mathbf{Q}^3$  of ternary form      **qfparam**( $G, x$ )  
 $P, Q \in \mathbf{F}_q[X]$ ; char. poly. of Frobenius      **hyperellcharpoly**( $Q$ )  
matrix of Frobenius on  $\mathbf{Q}_p \otimes H_{dR}^1$       **hyperellpadicfrobenius**

## Elliptic & Modular Functions

$w = [\omega_1, \omega_2]$  or *ell* struct (**E.omega**),  $\tau = \omega_1/\omega_2$ .  
arithmetic-geometric mean      **agm**( $x, y$ )  
elliptic  $j$ -function  $1/q + 744 + \dots$       **ellj**( $x$ )  
Weierstrass  $\sigma/\wp/\zeta$  function      **ellsigma**( $w, z$ ), **ellwp**, **ellzeta**  
periods/quasi-periods      **ellperiods**( $E, \{flag\}$ ), **elleta**( $w$ )  
 $(2i\pi/\omega_2)^k E_k(\tau)$       **elleisnum**( $w, k, \{flag\}$ )  
modified Dedekind  $\eta$  func.  $\prod(1 - q^n)$       **eta**( $x, \{flag\}$ )  
Dedekind sum  $s(h, k)$       **sumdedekind**( $h, k$ )  
Jacobi sine theta function      **theta**( $q, z$ )  
 $k$ -th derivative at  $z=0$  of **theta**( $q, z$ )      **thetanullk**( $q, k$ )  
Weber's  $f$  functions      **weber**( $x, \{flag\}$ )  
modular pol. of level  $N$       **polmodular**( $N, \{inv = j\}$ )  
Hilbert class polynomial for  $\mathbf{Q}(\sqrt{D})$       **polclass**( $D, \{inv = j\}$ )

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Send comments and corrections to (Karim.Belabas@math.u-bordeaux.fr)